NAMIBIA UNIVERSITY
OF SCIENCE AND TECHNOLOGY

# FACULTY OF HEALTH, APPLIED SCIENCES AND NATURAL RESOURCES

## DEPARTMENT OF MATHEMATICS AND STATISTICS

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| **QUALIFICATION:** Bachelor of Science in Applied Mathematics Honours | |
| **QUALIFICATION CODE:** 08BSHM | **LEVEL:** 8 |
| **COURSE CODE:** FAN802S | **COURSE NAME:** FUNCTIONAL ANALYSIS |
| **SESSION:** JANUARY 2023 | **PAPER:** THEORY |
| **DURATION:** 3H00 | **MARKS:** 100 |

| SECOND OPPORTUNITY/SUPPLEMENTARY -- QUESTION PAPER | |
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| **EXAMINER** | Dr S.N. NEOSSI NGUETCHUE |
| **MODERATOR:** | Prof F. MASSAMBA |

## INSTRUCTIONS

1. Answer ALL the questions in the booklet provided.
2. Show clearly all the steps used in proofs and obtaining results.
3. All written work must be done in blue or black ink and sketches must be done in pencil.

## PERMISSIBLE MATERIALS

1. Non-programmable calculator without a cover.

**THIS QUESTION PAPER CONSISTS OF 2 PAGES** (Including this front page)

**Attachments**

None

**Problem 1:** [45 Marks]

**1-1.** Let $f\colon \mathbb{R} \to \mathbb{R}$ such that $x \mapsto \begin{cases} 0, & \text{if } x \in \mathbb{Q}, \\ 1, & \text{if } x \notin \mathbb{Q}. \end{cases}$ Show that $f$ is Borel-measurable. [10]

(Hint: for any $a \in \mathbb{R}$, consider $E = \{x \in \mathbb{R} \colon f(x) < a\}$ and show that $f^{-1}(E) \in \mathcal{B}(\mathbb{R})$)

**1-2.** Let $(\mathbf{X}, \mathcal{F})$ be a measurable space. Prove that if $A_n \in \mathcal{F}, n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$. [5]

**1-3.** Let $\Omega$ be a non-empty set and $\mathcal{F}_\alpha \subset \mathcal{P}(\Omega), \alpha \in I$ an arbitrary collection of $\sigma$-algebras on $\Omega$. State the definition of a $\sigma$-algebra and prove that [4+6=10]

$$\mathcal{F} := \bigcap_{\alpha \in I} \mathcal{F}_\alpha \quad \text{is a } \sigma\text{-algebra.}$$

**1-4.** Let $(\mathbf{X}, \mathcal{A}, \mu)$ be a measure space.

(i) What does it mean that $(\mathbf{X}, \mathcal{A}, \mu)$ be a measure space? [3]

(ii) Show that for any $A, B \in \mathcal{A}$, we have the equality: $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$. [7]

(Hint: Consider two cases: (i) $\mu(A) = \infty$ or $\mu(B) = \infty$; (ii) $\mu(A), \mu(B) < \infty$ and then express $A, B, A \cup B$ in terms of $A \setminus B, B \setminus A, A \cap B$ where necessary.)

**1-5.** Show that the following Dirichlet function is Lebesgue integrable but not Riemann integrable [10]

$$\chi := \mathbb{1}_{\mathbb{Q}\cap[0,1]} \colon [0,1] \to \mathbb{R}$$
$$x \mapsto \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$$

**Problem 2:** [20 Marks]

**2-1.** Define what is a compact set in a topological space. [3]

**2-2.** Show that $(0, 1]$ is not a compact set for usual topology of $\mathbb{R}$. [9]

**2-3.** Let $E$ be a Hausdorff topological space and $\{a_n\}_{n\in\mathbb{N}}$ a sequence of elements of $E$ converging to $a$. Show that $K = \{a_n | n \in \mathbb{N}\} \cup \{n\}$ is compact in $E$. [8]

**Problem 3:** [35 Marks]

**3-1.** Use the convexity of $x \mapsto e^x$ to prove the Arithmetic-Geometric Mean inequality: [5]

$$\forall x, y > 0, \text{ and } 0 < \lambda < 1, \text{ we have: } x^\lambda y^{1-\lambda} \leq \lambda x + (1-\lambda)y.$$

**3-2.** Use the inequality in question **2-1.** to prove **Young's inequality**: [6]

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}, \ \forall \alpha, \beta > 0, \text{ where } p, q \in (1, \infty)\colon \frac{1}{p} + \frac{1}{q} = 1.$$

**3-3.** Use the result in question **3-2.** to prove Hölder's inequality: [7]

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}, \forall\, \mathbf{x} = (x_i), \mathbf{y} = (y_i) \in \mathbb{R}^n, \ p, q \text{ as above}.$$

**3-4.** Consider $(\mathbf{X}, \|\cdot\|_{\infty,1})$, where $\mathbf{X} = \mathcal{C}^1[0,1]$ and $\|f\|_{\infty,1} = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|$ and also consider $(\mathbf{Y}, \|\cdot\|_\infty)$, where $\mathbf{Y} = \mathcal{C}[0,1]$.

**3-4-1.** Show that $T = \dfrac{d}{dx} \colon \mathbf{X} \to \mathbf{Y}$ is a bounded linear operator. [7]

**3-4-2.** Show that $T = \dfrac{d}{dx} \colon D(T) \subsetneq \mathbf{Y} \to \mathbf{Y}$ is an unbounded linear operator, where $D(T) = \mathcal{C}^1[0,1]$. [10]

(Hint: use $u_n(x) = \sin(n\pi x)$).

God bless you !!!